Variational problems of nonlinear elasticity in certain classes of mappings with finite distortion *

A. O. Molchanova, S. K. Vodop′yanov

Contents

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Abstract

We study the problem of minimizing the functional

$$I(\varphi) = \int\limits_{\Omega} W(x, D\varphi)\, dx$$

on a new class of mappings. We relax summability conditions for admissible deformations to $\varphi \in W^1_n(\Omega)$ and growth conditions on the integrand $W(x, F)$. To compensate for that, we impose the finite distortion condition and the condition $\frac{|D\varphi(x)|^n}{J(x,\varphi)} \leq M(x) \in L_s(\Omega)$, $s > n-1$, on the characteristic of distortion. On assuming that the integrand $W(x, F)$ is polyconvex and coercive, we obtain an existence theorem for the problem of minimizing the functional $I(\varphi)$ on a new family of admissible deformations.
Keywords: functional minimization problem, nonlinear elasticity, mapping with finite distortion, polyconvexity.

1 Introduction

Some problems in nonlinear elasticity (for instance, for hyperelastic materials) reduce to minimizing the total energy functional. In this situation, in contrast to the case of linear elasticity, the integrand is almost always nonconvex, while the functional is nonquadratic. This renders the standard variational methods inapplicable. Nevertheless, for a sufficiently large class of applied nonlinear problems, we may replace convexity with certain weaker conditions. In 1952 Charles Morrey suggested to consider *quasiconvex* functions (see [19] for more details). Denote by $\mathbb{M}^{m\times n}$ the set of $m \times n$ matrices. A bounded measurable function $f : \mathbb{M}^{m\times n} \to \mathbb{R}\cup\{\infty\}$ is called *quasiconvex* if

$$\int\limits_{\Omega} f(F_0 + D\zeta(x))\, dx \geq \int\limits_{\Omega} f(F_0)\, dx = |\Omega| \cdot f(F_0). \tag{1.1}$$

holds for each constant $m \times n$ matrix F_0 for each bounded open set $\Omega \subset \mathbb{R}^n$ and for all $\zeta \in C^\infty_0(\Omega, \mathbb{R}^m)$.

Morrey showed that if $W(\cdot, \cdot)$ is quasiconvex and satisfies certain smoothness and growth conditions then the problem of minimizing the total energy

functional

$$I(\varphi) = \int_{\Omega} W(x, D\varphi) + \Theta(x, \varphi)\, dx \tag{1.2}$$

has a solution, where $W(x, D\varphi)$ is the stored-energy function and $\Theta(x, \varphi)$ is a body force potential. Conversely, if there exists a mapping minimizing the functional in the class $C^1(\overline{\Omega})$ functions satisfying given boundary conditions, then the quasiconvexity condition (1.1) is fulfilled. Although Morrey's results are significant for the theory, the conditions imposed turn out too restrictive, excluding applications to an important class of nonlinear elasticity problems.

In 1977 John Ball developed another successful approach to nonlinear elasticity problems using the concept of *polyconvexity* (see [2] for more details). Even though every polyconvex function is also quasiconvex, weaker growth conditions than in Morrey's articles are used to prove the existence theorem.

Ball's method is to consider a sequence $\{\varphi_k\}$ minimizing the total energy functional (1.2) over the set of admissible deformations

$$\begin{aligned} \mathcal{A}_B = \{\varphi \in W_1^1(\Omega), I(\varphi) < \infty,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma \text{ a. e. in } \Gamma = \partial\Omega, \\ J(x, \varphi) > 0 \text{ a. e. in } \Omega\}, \end{aligned} \tag{1.3}$$

where $\overline{\varphi}$ are Dirichlet boundary conditions, on assuming that the *coercivity inequality*

$$W(x, F) \geq \alpha(\|F\|^p + \|\operatorname{Adj} F\|^q + (\det F)^r) + g(x) \tag{1.4}$$

holds for almost all $x \in \Omega$ and all $F \in \mathbb{M}^n_+$, where $p > n-1$, $q \geq \frac{p}{p-1}$, $r > 1$ and $g \in L_1(\Omega)$, while $\mathbb{M}^n_+$ stands for the set of matrices of size $n \times n$ with positive determinant. Moreover, the stored-energy function W is *polyconvex*, that is, there exists a convex function $G(x, \cdot) : \mathbb{M}^n \times \mathbb{M}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$G(x, F, \operatorname{Adj} F, \det F) = W(x, F) \text{ for all } F \in \mathbb{M}^n_+$$

almost everywhere in Ω. By coercivity, the sequence $(\varphi_k, \operatorname{Adj} D\varphi_k, J(\varphi_k))$ is bounded in the reflexive Banach space $W^1_p(\Omega) \times L_q(\Omega) \times L_r(\Omega)$. Hence, there exists a subsequence weakly converging to an element $(\varphi_0, \operatorname{Adj} D\varphi_0, J(\varphi_0))$. For the limit φ_0 to lie in the class $\mathcal{A}_B$ of admissible deformations, we need to impose the additional condition:

$$W(x, F) \to \infty \text{ as } \det F \to 0_+ \tag{1.5}$$

(see [4] for more details). This condition is quite reasonable since it fits in with the principle that *"infinite stress must accompany extreme strains"*. Another important property in this approach is the *sequentially weak lower semicontinuity* of the total energy functional,

$$I(\varphi) \le \varliminf_{k\to\infty} I(\varphi_k),$$

which holds because the stored-energy function is polyconvex. It is also worth noting that Ball's approach admits the *nonuniqueness of solutions* observed experimentally (see [2] for more details).

Philippe Ciarlet and Indrich Nečas studied [8] injective deformations, imposing the *injectivity condition*

$$\int\limits_{\Omega} J(x,\varphi)\,dx \le |\varphi(\Omega)|$$

on the admissible deformations and requiring extra regularity ($p > n$). Under these assumptions, there exists an almost everywhere injective minimizer of the total energy functional.

In the case of problems with boundary conditions on displacement the injectivity condition turns out superfluous when the deformation on the boundary coincides with a homeomorphism and the stored-energy function W tends to infty sufficiently fast. Ball obtained this result in 1981 [3] (a misprint is corrected in Exercise 7.13 of [7]). More exactly, take a domain $\Omega \subset \mathbb{R}^3$ and a polyconvex stored-energy function $W : \Omega \times \mathbb{M}^3_+ \to \mathbb{R}$. Suppose that there exist constants $\alpha > 0$, $p > 3$, $q > 3$, $r > 1$, and $m > \frac{2q}{q-3}$, as well as a function $g \in L_1(\Omega)$ such that

$$W(x,F) \ge \alpha(\|F\|^p + \|\operatorname{Adj} F\|^q + (\det F)^r + (\det F)^{-m}) + g(x) \qquad (1.6)$$

for almost all $x \in \Omega$ and all $F \in \mathbb{M}^n_+$. Take a homeomorphism $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ in $W^1_p(\Omega)$. Then there exists a mapping $\varphi : \Omega \to \Omega'$ minimizing the total energy functional (1.2) over the set of admissible deformations (1.3), which is a homeomorphism.

We should note that the above conditions on the adjoint matrix $\operatorname{Adj} Df \in L_q(\Omega)$, with $q > 3$, and $\frac{1}{\det Df} \in L_m(\Omega)$, where $m > \frac{2q}{q-3}$, in fact constrain the inverse mapping (since $Df^{-1} = \frac{\operatorname{Adj} Df}{\det Df}$). In this article we discard con-

straints on the inverse mapping and consider a new class of admissible deformations:

$$\mathcal{A} = \{\varphi \in W_1^1(\Omega) \cap FD(\Omega),\ I(\varphi) < \infty,\ \frac{|D\varphi(x)|^n}{J(x,\varphi)} < M(x) \in L_s(\Omega),$$
$$s > n-1,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma \text{ a. e. in } \Gamma,\ J(x,\varphi) \geq 0 \text{ a. e. in } \Omega\}, \quad (1.7)$$

where $FD(\Omega)$ is the class of mappings with finite distortion. Another significant difference is the weakening of conditions on the stored-energy function. The coercivity inequality becomes

$$W(x,F) \geq \alpha(\|F\|^n + (\det F)^r) + g(x). \quad (1.8)$$

In some previous works the deformation φ was required to lie in the Sobolev class $W_p^1(\Omega)$ with $p > n$ when there is a compact embedding of $W_p^1(\Omega)$ into the space of continuous functions $C(\Omega)$, that is, assume at the outset that φ is continuous. In this article we only assume that $\varphi \in W_n^1(\Omega)$; consequently, we must prove separately that the admissible deformation is continuous. The second feature of this article is the replacement of conditions on the "inverse mapping" by the summability of the distortion coefficient $\frac{|D\psi(x)|^n}{J(x,\psi)} < M(x) \in L_s(\Omega)$, where $s > n-1$.

The main result of this article is the following theorem.

Theorem 1.1. *Given a polyconvex function $W(x,F)$ satisfying the coercivity inequality* (1.8)*, a homeomorphism $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$, $\overline{\varphi} \in W_n^1(\Omega)$, and a nonempty set $\mathcal{A}$, there exists at least one mapping $\varphi_0 \in \mathcal{A}$ such that $I(\varphi_0) = \inf\limits_{\varphi \in \mathcal{A}} I(\varphi)$. Moreover, $\varphi_0 : \overline{\Omega} \to \overline{\Omega'}$ is a homeomorphism.*

It is not difficult to verify that for sufficiently large values of the exponents $q > n(n-1)s$ and $m > \frac{sq(n-1)^2}{q-sn(n-1)}$ the summability of the distortion coefficient $\frac{|D\psi(x)|^n}{J(x,\psi)} \in L_s(\Omega)$ of the mapping $\psi \in \mathcal{A}$ follows from (1.6) and Corollary 6 of [33]. We should note also that in this article the property of mapping to be sense preserving follows from the property that the required deformation is a mapping with bounded (n,q)-distortion [5].

Naturally, the proofs of our main results differ substantially from Ball's methods in [2, 3] and depend crucially on the results and methods of [33].

In the first section we present the concept of polyconvexity of functions. The second section contains auxiliary facts. The third section is devoted to

the main result: the existence theorem and its proof. In the fourth section we give two examples. In the first example we consider a stored-energy function $W(F)$ for which both minimization problems (in the classes $\mathcal{A}_B$ and $\mathcal{A}$) have solutions. The second example discusses a function $W(F)$ violating the coercivity (1.4) and asymptotic condition (1.5); nevertheless, there exists a solution to the minimization problem in $\mathcal{A}$.

The results of this article were announced in the note [36] together with a sketch of the proof of the main result.

2 The concept of polyconvexity

For a large class of physical problems it may be assumed that the stored-energy function is *polyconvex*.

Definition 1 ([2]). A function $W : \mathbb{F} \to \mathbb{R}$ defined on an arbitrary subset $\mathbb{F} \subset \mathbb{M}^3$ is *polyconvex* if there exists a convex function $G : \mathbb{U} \to \mathbb{R}$, where

$$\mathbb{U} = \{(F, \operatorname{Adj} F, \det F) \in \mathbb{M}^3 \times \mathbb{M}^3 \times \mathbb{R},\ F \in \mathbb{F}\},$$

such that

$$G(F, \operatorname{Adj} F, \det F) = W(F) \text{ for all } F \in \mathbb{F}.$$

Here $\operatorname{Adj} F$ stands for the adjugate matrix, that is, the transpose of cofactor matrix.

As examples of polyconvex but not convex functions, consider

$$W(F) = \det F$$

and

$$W(F) = \operatorname{tr} \operatorname{Adj} F^T F = \| \operatorname{Adj} F^T F\|^2.$$

2.1 Polyconvex stored-energy functions

Consider the stored-energy function

$$W(F) = \sum_{i=1}^{M} a_i \operatorname{tr} (F^T F)^{\frac{\gamma_i}{2}} + \sum_{j=1}^{N} b_j \operatorname{Adj}(F^T F)^{\frac{\delta_j}{2}} + \Gamma(\det F)$$

with $a_i > 0$, $b_j > 0$, $\gamma_i \geq 1$, $\delta_j \geq 1$, and a convex function $\Gamma : (0, \infty) \to \mathbb{R}$. If

$$\lim_{\delta \to 0_+} \Gamma(\delta) = \infty$$

then the hyperelastic material is called an *Ogden material* [21]. These materials have polyconvex stored-energy function and satisfy the growth conditions of Ball's existence theorem.

Ogden materials are interesting not only in theory, but also in practice. Moreover (see [7] for more details), for a hyperelastic material with experimentally known Lamé coefficients it can be constructed a stored-energy function of an Ogden material.

2.2 Non-polyconvex stored-energy functions

Saint-Venant–Kirhhoff materials are well-known examples of hyperelastic materials. Its stored-energy function is

$$W(F) = \frac{\lambda}{2} (\operatorname{tr} E)^2 + \mu \operatorname{tr} E^2,$$

where λ and μ are Lamé coefficients and

$$I + 2E = F^T F.$$

This function is a particular case of the function

$$W(F) = a_1 \operatorname{tr} F^T F + a_2 \operatorname{tr} (F^T F)^2 + b \operatorname{tr} \operatorname{Adj} F^T F + c$$

with $a_1 < 0$, $a_2 > 0$, and $b > 0$. Although this function resembles the function of an Ogden material and satisfies the coercivity inequality, it is not polyconvex [7, Theorem 4.10].

3 Preliminaries

In this section we present some important concepts and statements necessary to proceed. On a domain $\Omega \subset \mathbb{R}^n$ we define in a standard way (see [18] for instance) the spaces $C_0^\infty(\Omega)$ of smooth functions with compact support, the Lebesgue spaces $L_p(\Omega)$ and $L_{p,\mathrm{loc}}(\Omega)$ of integrable functions, and the Sobolev spaces $W_p^1(\Omega)$ and $W_{p,\mathrm{loc}}^1(\Omega)$.

For working with a geometry of domains we need the following definition.

Definition 2. A homeomorphism $\varphi : \Omega \to \Omega'$ of two open sets Ω, $\Omega' \subset \mathbb{R}^n$ is called a *quasi-isometric mapping* if the following inequalities

$$\overline{\lim_{y\to x}} \frac{|\varphi(y)-\varphi(x)|}{|y-x|} \le M \quad \text{and} \quad \overline{\lim_{y\to z}} \frac{|\varphi^{-1}(y)-\varphi^{-1}(z)|}{|y-z|} \le M$$

hold for all $x \in \Omega$ and $z \in \Omega'$ where M is some constant independent of the choice of points $x \in \Omega$ and $z \in \Omega'$.

Recall the following definition.

Definition 3. A domain $\Omega \subset \mathbb{R}^n$ is called a domain with *locally quasi-isometric boundary* whenever for every point $x \in \partial\Omega$ there are a neighborhood $U_x \subset \mathbb{R}^n$ and a quasi-isometric mapping $\nu_x : U_x \to B(0, r_x) \subset \mathbb{R}^n$, where the number $r_x > 0$ depends on U_x, such that $\nu(U_x \cap \partial\Omega) \subset \{y \in B(0, r_x) \mid y_n = 0\}$.

Remark 3.1. In some papers it is used a bi-Lipschitz mapping φ instead of quasi-isometric mapping in this definition. It is evident that the bi-Lipschitz mapping is also quasi-isometric one. The inverse implication is not valid but it is valid the following assertion: *every quasi-isometric mapping is locally bi-Lipschitz one* (see a proof below). Hence Ω is a domain with locally Lipschitz boundary if and only if it is a domain with quasi-isometric boundary.

Proof. For proving this statement fix a quasi-isometric mapping $\varphi : \Omega \to \Omega'$. We have to verify for any fixed ball $B \Subset \Omega$ the inequality

$$d_{\varphi(B)}(\varphi(x), \varphi(y)) \le L|\varphi(x) - \varphi(y)|$$

holds for all points x, $y \in B$ with some constant L depending on the choice of B only (here $d_{\varphi(B)}(u, v)$ is the intrinsic metric in the domain $\varphi(B)$ defined as the infimum over the lengths of all rectifiable curves in $\varphi(B)$ with endpoints u and v)[1]. Take an arbitrary function $g \in W^1_\infty(\varphi(B))$. Then $\varphi^*(g) = g \circ \varphi \in W^1_\infty(B)$ and by Whitney type extension theorem (see for instance [28, 29]) there is a bounded extension operator $\mathrm{ext}_B : W^1_\infty(B) \to W^1_\infty(\mathbb{R}^n)$. Multiply $\mathrm{ext}_B(\varphi^*(g))$ by a cut-of-function $\eta \in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ for all

[1] It is well-known that a mapping is quasi-isometric iff the lengths of a rectifiable curve in the domain and of its image are comparable. The last property is equivalent to the following one: given mapping $\varphi : \Omega \to \Omega'$ is quasi-isometric iff $L^{-1} d_B(x, y) \le d_{\varphi(B)}(\varphi(x), \varphi(y)) \le L d_B(x, y)$ for all $x, y \in B$.

points $x \in B$. Then the product $\eta \cdot \mathrm{ext}_B(\varphi^*(g))$ belongs to $W^1_\infty(\Omega)$, equals 0 near the boundary $\partial\Omega$ and its norm in $W^1_\infty(\Omega)$ is controlled by the norm $\|g \mid W^1_\infty(\varphi(B))\|$.

It is clear that $\varphi^{-1*}(\eta \cdot \mathrm{ext}_B(\varphi^*(g)))$ belongs to $W^1_\infty(\Omega')$, equals 0 near the boundary $\partial\Omega'$ and its norm in $W^1_\infty(\Omega')$ is controlled by the norm $\|g \mid W^1_\infty(\varphi(B))\|$. Extending $\varphi^{-1*}(\eta \cdot \mathrm{ext}_B(\varphi^*(g)))$ by 0 outside Ω' we obtain a bounded extension operator

$$\mathrm{ext}_{\varphi(B)} : W^1_\infty(\varphi(B)) \to W^1_\infty(\mathbb{R}^n).$$

It is well-known (see for example [28, 29]) that a necessary and sufficient condition for existence of such operator is an equivalence of the interior metric in $\varphi(B)$ to the Euclidean one: the inequality

$$d_{\varphi(B)}(u, v) \leq L|u - v|$$

holds for all points u, $v \in \varphi(B)$ with some constant L. □

Taking into account Remark 3.1 we can consider also a domain with quasi-isometric boundary instead of domain with Lipschitz boundary in the statements formulated below.

Theorem 3.1 (Rellich–Kondrachov theorem, see [1] for instance)**.** *Consider a bounded domain Ω in $\mathbb{R}^n$ and $1 \leq p < \infty$. If Ω satisfies the cone condition then the following embeddings are compact:*

1. $W^1_p(\Omega) \Subset L_q(\Omega)$ *for* $1 \leq q < p^* = \frac{np}{n-p}$ *with* $p < n$*;*
2. $W^1_p(\Omega) \Subset L_q(\Omega)$ *for* $1 \leq q < \infty$ *with* $p = n$*.*
3. *If Ω has a locally Lipschitz boundary $\partial\Omega$ then for $p > n$ the embedding $W^1_p(\Omega) \Subset C(\overline{\Omega})$ is compact.*

Theorem 3.2 (Properties of the trace operator [18])**.** *Consider a bounded domain Ω with locally Lipschitz boundary $\partial\Omega$ endowed with the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ and $1 \leq p < \infty$. There exists a bounded linear operator* tr *such that* $\mathrm{tr}\, f = f$ *on $\partial\Omega$ for all $f \in W^1_p(\Omega) \cap C(\overline{\Omega})$, with properties:*

1. *if $1 \leq p < n$ then* $\mathrm{tr} : W^l_p(\Omega) \to L_q(\partial\Omega)$ *for $1 < q < p^* = \frac{(n-1)p}{n-p}$ and furthermore, for $1 < p < n$ the operator* tr *is compact;*

2. *if* $p = n$ *then* $\operatorname{tr} : W_p^l(\Omega) \to L_q(\partial\Omega)$ *for* $1 < q < \infty$ *and furthermore, the operator* tr *is compact;*

3. *if* $n < p$ *then* $\operatorname{tr} : W_p^l(\Omega) \to C(\partial\Omega)$. *and furthermore, the operator* tr *is compact.*

We also need the following theorem.

Theorem 3.3 (Corollary to Poincaré inequality, see [7] for instance). *Given a connected bounded domain* $\Omega \subset \mathbb{R}^n$ *with Lipschitz boundary* $\Gamma = \partial\Omega$, *a measurable subset* Γ_0 *of* Γ *with* $|\Gamma_0| > 0$, *and* $1 \leq p < \infty$, *there exists a constant* C_1 *such that*

$$\int\limits_{\Omega} |f(x)|^p\,dx \leq C_1\biggl(\int\limits_{\Omega} |Df(x)|^p\,dx + \biggl|\int\limits_{\Gamma_0} f(x)\,ds\biggr|^p\biggr).$$

for all $f \in W_p^1(\Omega)$.

Lemma 3.1 ([23, 6]). *Take an open connected set* $\Omega \subset \mathbb{R}^n$ *with* $n \geq 2$ *and a mapping* $f : \Omega \to \mathbb{R}^n$ *with* $f \in W_p^1(\Omega)$, *for* $p > n-1$. *Then the columns of the matrix* $\operatorname{Adj} Df(x)$ *are divergence-free vector fields, that is,* $\sum\limits_{j=1}^{n} \frac{\partial}{\partial x_j} A_{jk} = 0$ *in the sense of distributions for all* $k = 1, \dots, n$.

Remark 3.2. The proofs in [23, 6] rest on smooth approximations to a Sobolev mapping. The new proof of this lemma in [32] avoids these approximations.

We also need the following corollary to the Bezikovich theorem (see Theorem 1.1 in [12] for instance).

Lemma 3.2. *For every open set* $U \subset \mathbb{R}^n$ *with* $U \neq \mathbb{R}^n$ *there exists a countable family* $\mathcal{B} = \{B_j\}$ *of balls such that*

1. $\bigcup\limits_j B_j = U$;

2. *if* $B_j = B_j(x_j, r_j) \in \mathcal{B}$ *then* $\operatorname{dist}(x_j, \partial U) = 12r_j$;

3. *the families* $\mathcal{B} = \{B_j\}$ *and* $2\mathcal{B} = \{2B_j\}$, *where the symbol* $2B$ *stands for the ball of doubled radius centered at the same point, constitute a finite covering of* U;

4. *if the balls* $2B_j = B_j(x_j, 2r_j)$, $j = 1, 2$ *intersect then* $\frac{5}{7}r_1 \le r_2 \le \frac{7}{5}r_1$*;*
5. *we can subdivide the family* $\{2B_j\}$ *into finitely many tuples so that in each tuple the balls are disjoint and the number of tuples depends only on the dimension* n.

The main concepts of functional analysis like weak convergence, semicontinuous functionals, and related theorems are described in detail in [10, 16]. Let us recall some of them.

Theorem 3.4. *For a normed vector space* V *and a Banach space* W *consider a continuous bilinear mapping* $B : V \times W \to \mathbb{R}$. *If* $v_k \to v$ *strongly and* $w_k \to w$ *weakly then* $B(v_k, w_k) \to B(v, w)$.

Theorem 3.5 (Mazur theorem, see [10] for instance)**.** *Let* $v_k \to v$ *weakly in a normed vector space* V. *Then there exist convex combinations*

$$w_k = \sum_{m=k}^{N(k)} \lambda_m^k v_k, \quad \textit{where} \quad \sum_{m=k}^{N(k)} \lambda_m^k = 1, \quad \lambda_m^k \ge 0, \quad k \le m \le N(k),$$

converging to v *in norm.*

Definition 4. A function $J : V \to \mathbb{R} \cup \{\infty\}$ is called *sequentially weakly lower semicontinuous* whenever

$$J(u_0) \le \varliminf_{u_k \to u_0} J(u_k)$$

for every weakly converging sequence $\{u_k\} \subset V$.

Definition 5. Let an open set $\Omega \subset \mathbb{R}^k$, a mapping $G : \Omega \times \mathbb{R}^m \to \overline{\mathbb{R}}$ enjoys the *Carathéodory conditions* whenever

(a) $G(x, \cdot)$ is continuous on $\mathbb{R}^m$ for almost all $x \in \Omega$;
(b) $G(\cdot, a)$ is measurable on Ω for all $a \in \mathbb{R}^m$.

The reader not familiar with mappings with bounded distortion may look at [25, 23]. Let us recall the main concepts and theorems.

Theorem 3.6 ([23])**.** *Consider a sequence* $f_m : \Omega \to \mathbb{R}^n$ *of mappings of class* $W^1_{n,\mathrm{loc}}(\Omega)$. *Suppose that the following conditions are met:*

(a) $\{f_m\}$ *is locally bounded in* $W^1_{n,\mathrm{loc}}(\Omega)$*;*

(b) $\{f_m\}$ *converges in* $L_1(\Omega)$ *to some mapping* f_0 *as* $m \to \infty$.
Then $f_0 \in W^1_{n,\mathrm{loc}}(\Omega)$ *and*

$$\int\limits_{\Omega} \varphi(x) J(x, f_m)\, dx \longrightarrow \int\limits_{\Omega} \varphi(x) J(x, f_0)\, dx \qquad \text{as } m \longrightarrow \infty$$

for every continuous real function $\varphi : U \to \mathbb{R}$ *with compact support in* Ω.

Definition 6 ([14])**.** A mapping $f : \Omega \to \mathbb{R}^n$ with $f \in W^1_{1,\mathrm{loc}}(\Omega)$ is called a *mapping with finite distortion*, written $f \in FD(\Omega)$, whenever $J(x, f) \geq 0$ almost everywhere in Ω and

$$|Df(x)|^n \leq K(x) J(x, f) \quad \text{for almost all } x \in \Omega,$$

where $0 < K(x) < \infty$ almost everywhere in Ω.

Remark 3.3. In other words, the *finite distortion* condition amounts to the vanishing of the partial derivatives of $f \in W^1_{1,\mathrm{loc}}(\Omega)$ almost everywhere on the zero set of the Jacobian.

Definition 7 ([23])**.** Given $f : \Omega \to \mathbb{R}^n$ with $f \in W^1_{n,\mathrm{loc}}(\Omega)$ and $J(x, f) \geq 0$ almost everywhere in Ω, the *distortion* of f at x is the function

$$K(x) = \begin{cases} \frac{|Df(x)|^n}{J(x,f)} & \text{if } J(x, f) > 0, \\ 1 & \text{if } J(x, f) = 0. \end{cases}$$

Theorem 3.7 ([17])**.** *Let* $f \in W^1_{n,\mathrm{loc}}(\Omega)$ *be nonconstant mapping whose dilatation* $K(x)$ *is in* $L_{s,\mathrm{loc}}(\Omega)$*. Then, if* $s > n - 1$ *and* $J(x, f) \geq 0$ *for almost all* $x \in \Omega$*, the mapping* f *is continuous, discrete, and open.*

Remark 3.4. Theorem 2.3 of [34] shows that this mapping is continuous.

Definition 8 ([31])**.** *a mapping* $\varphi : \Omega \to \Omega'$ *induces a bounded operator* $\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$ *by the composition rule*, $1 \leq q \leq p < \infty$, if the following properties fulfil:

(a) If two quasicontinuous functions f_1, $f_2 \in L^1_p(\Omega')$ are distinguished on a set of p-capacity zero, then the functions $f_1 \circ \varphi$, $f_2 \circ \varphi$ are distinguished on a set of measure zero;

(b) If $\tilde{f} \in L^1_p(\Omega')$ is a quasicontinuous representative of f, then $\tilde{f} \circ \varphi \in L^1_p(\Omega)$ but $\tilde{f} \circ \varphi$ is not required to be quasicontinuous;

(c) The mapping $\varphi^* : f \mapsto \tilde{f} \circ \varphi$, where $\tilde{f}$ is a quasicontinuous representative of f, is a bounded operator $L^1_p(\Omega') \to L^1_q(\Omega)$.

The definition and properties p-capacity see, for instance, in [9, 18, 22].

Theorem 3.8 ([38, Theorem 4]). *Take two open sets Ω and Ω' in $\mathbb{R}^n$ with $n \geq 1$. If a mapping $\varphi : \Omega \to \Omega'$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\Omega') \cap C^\infty(\Omega') \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq n,$$

with $\varphi(f) = f \circ \varphi$ then φ has Luzin $\mathcal{N}^{-1}$-property.

Remark 3.5. Theorem 4 of [38] is stated for a mapping $\varphi : \Omega \to \Omega'$ generating a bounded composition operator $\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$ with $1 \leq q \leq p \leq n$. Observe that only smooth test functions are used in its proof, which therefore also justifies Theorem 3.8.

Following [33], for a mapping $f : \Omega \to \Omega'$ of class $W^1_{1,\mathrm{loc}}(\Omega)$ define the *distortion operator function*

$$K_{f,p}(x) = \begin{cases} \frac{|Df(x)|}{|J(x,f)|^{\frac{1}{p}}} & \text{for } x \in \Omega \setminus (Z \cap \Sigma), \\ 0 & \text{otherwise,} \end{cases}$$

where Z is the zero set of the Jacobian $J(x, f)$ and Σ is a singularity set, meaning that $|\Sigma| = 0$ and f enjoys Luzin $\mathcal{N}$-property outside Σ.

Theorem 3.9 ([33, 37, 38]). *A homeomorphism $\varphi : \Omega \to \Omega'$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega), \quad 1 \leq q \leq p < \infty,$$

where $\varphi^*(f) = f \circ \varphi$ for $f \in L^1_p(\Omega')$, if and only if the following conditions are met:

1. *$\varphi \in W^1_{q,\mathrm{loc}}(\Omega)$;*
2. *the mapping φ has finite distortion;*
3. *$K_{\varphi,p}(\cdot) \in L_\varkappa(\Omega)$, where $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$, $1 \leq q \leq p < \infty$ (and $\varkappa = \infty$ for $q = p$).*

Moreover,

$$\|\varphi^*\| \leq \|K_{\varphi,p}(\cdot) \mid L_\varkappa(\Omega)\| \leq C\|\varphi^*\|$$

for some constant C.

Remark 3.6. Necessity is proved in [37, 38] (see also earlier work [26]), and sufficiency, in Theorem 5 of [33].

Theorem 3.10 ([33])**.** *Assume that a homeomorphism $\varphi : \Omega \to \Omega'$*

1. *induces a bounded composition operator $\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$ for $n-1 \leq q \leq p \leq \infty$, where $\varphi^*(f) = f \circ \varphi$ for $f \in L^1_p(\Omega')$,*
2. *has finite distortion for $n-1 \leq q \leq p = \infty$.*

Then the inverse mapping φ^{-1} induces a bounded composition operator $\varphi^{-1*} : L^1_{q'}(\Omega) \to L^1_{p'}(\Omega')$, where $q' = \frac{q}{q-n+1}$ and $p' = \frac{p}{p-n+1}$, and has finite distortion.

Moreover,

$$\|\varphi^{-1*}\| \leq \|K_{\varphi^{-1},q'}(\cdot) \mid L_\rho(\Omega')\| \leq \|K_{\varphi,p}(\cdot) \mid L_\varkappa(\Omega)\|^{n-1}, \text{ where } \frac{1}{\rho} = \frac{1}{p'} - \frac{1}{q'}.$$

Definition 9. Consider $f : \Omega \to \mathbb{R}^n$ and $D \subset \Omega$. The function $N_f(\cdot, D) : \mathbb{R}^n \to \mathbb{N} \cup \{\infty\}$ defined as

$$N_f(y, D) = \operatorname{card}(f^{-1}(y) \cap D)$$

is called the *Banach indicatrix.*

Theorem 3.11 (Change-of-variable formula[13, Theorem 2])**.** *Given an open set $\Omega \subset \mathbb{R}^n$, if a mapping $f : \Omega \to \mathbb{R}^n$ is approximatively differentiable almost everywhere on Ω then we can redefine f on a negligible set to gain Luzin $\mathcal{N}$-property.*

If a mapping $f : \Omega \to \mathbb{R}^n$ is approximatively differentiable almost everywhere on Ω and has Luzin $\mathcal{N}$-property then for every measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and every measurable set $D \subset \Omega$ we have:

1. *the functions $(u \circ f)(x)|J(x, f)|$ and $u(y)N_f(y, D)$ are measurable;*
2. *if $u \geq 0$ then*
$$\int\limits_D (u \circ f)(x)|J(x, f)|\, dx = \int\limits_{\mathbb{R}^n} u(y)N_f(y, D)\, dy; \tag{3.1}$$
3. *if one of the functions $(u \circ f)(x)|J(x, f)|$ and $u(y)N_f(y, D)$ is integrable then so is the second, and the change-of-variable formula (3.1) holds.*

Definition 10. Consider a continuous, open, and discrete mapping $f : \Omega \to \mathbb{R}^n$ and a point $x \in \Omega$. There exists the domain V containing x such that $\overline{V} \cup f^{-1}(f(\{x\})) = \{x\}$. Refer as the *local index* of f at x to the quantity $i(x, f) = \mu(f(x), f, V)$.

Remark 3.7. The local index is well-defined and independent of the domain V under consideration [25, 23].

Definition 11. Given a continuous mapping $f : \Omega \to \mathbb{R}^n$, say that f is *sense preserving* whenever the topological degree satisfies $\mu(y, f, D) > 0$ for every domain $D \Subset \Omega$ and every point $y \in f(D) \setminus f(\partial\Omega)$.

Definition 12. A mapping $f : \Omega \to \mathbb{R}^n$ is called a *mapping with bounded* (p, q)*-distortion*, with $n - 1 < q \leq p < \infty$, whenever

1. the mapping f is continuous, open, and discrete;
2. the mapping f is of Sobolev class $W^1_{q,\mathrm{loc}}(\Omega)$;
3. its Jacobian satisfies $J(x, f) \geq 0$ for almost all $x \in \Omega$;
4. the mapping f has finite distortion: $Df(x) = 0$ if and only if $J(x, f) = 0$ with the possible exclusion of a negligible set;
5. the local distortion function $K_{f,p}(x)$ belongs to $L_\varkappa(\Omega)$ where $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ ($\varkappa = \infty$ for $p = q$).

Lemma 3.3 ([5])**.** *A mapping $f : \Omega \to \mathbb{R}^n$ with bounded (p, q)-distortion is sense preserving in the case $q > n - 1$.*

Proof. Indeed, if f is a homeomorphism then $J(x, f) = 0$ cannot hold almost everywhere on Ω because this would imply that $Df(x) = 0$ almost everywhere, and so f would not be an open mapping. Consequently, $J(x, f) > 0$ on a set of positive measure and there exists a point $x \in \Omega$ of differentiability of f at which the differential is nondegenerate, while the Jacobian is positive (see [30, Proposition 1] for instance). The properties of degrees of mappings imply that f is sense preserving.

The general case reduces to the previous one since the image of the set of branch points is closed in $f(\Omega)$. Indeed, take $D \Subset \Omega$ avoiding the branch points and a point z in a connected component U_1 of the set $f(D) \setminus f(\partial D)$. Put $D_1 = f^{-1}(U_1) \subset D$. Since D_1 is an open set, the Jacobian $J(\cdot, f)$

cannot vanish almost everywhere on it (otherwise, D_1 would map to a point in contradiction with the openness of f). Hence, $J(x, f) > 0$ on a set of positive measure. For a point $x_0 \in D_1$ in this set we have $y = f(x_0) \in U_1$. Consider $f^{-1}(y) \cap D_1 = \{x_0, x_1, \ldots x_N\}$. Since f is a discrete mapping, there exists a ball $B(y, r)$ of small radius r with $f^{-1}(B(y, r)) = \cup W_j$, where $x_j \in W_j$ and $W_i \cap W_k = \emptyset$. Since $f : W_j \to B(y, r)$ is a homeomorphism, the local index $i(x_j, f) = \mu(y, f, W_j)$ is positive, while the degree satisfies

$$\mu(x, f, D_1) = \sum_{j=1}^{N} i(x_j, f) = \sum_{j=1}^{N} \mu(y, f, B_j) > 0,$$

see Proposition 4.4 of [25].

Now take $D \Subset \Omega$ intersecting the set V of branch points and suppose that the image of a branch point z lies in the connected component U_1 of $f(D) \setminus f(\partial D)$. Since the image of the set of branch points is closed, there must be points in U_1 which are outside of $f(V)$. Applying the previous argument to a point $z_1 \in U_1 \setminus f(V)$, we infer that $\mu(x, f, D_1) > 0$. Thus, the mapping f is sense preserving. $\square$

4 The Main Result

4.1 Existence theorem

Consider two bounded domains Ω, $\Omega' \subset \mathbb{R}^n$ with locally Lipschitz boundaries $\partial\Omega = \Gamma$ and $\partial\Omega' = \Gamma'$. Consider the functional

$$I(\varphi) = \int\limits_{\Omega} W(x, D\varphi(x))\, dx,$$

where $W : \Omega \times \mathbb{M}^n \to \mathbb{R}$ is a stored-energy function with the following properties:

(a) polyconvexity: there exists a convex function $G(x, \cdot) : \mathbb{M}^n \times \mathbb{M}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, satisfying Carathérode conditions, such that for all $F \in \mathbb{M}^n_{\geq 0}$ the equality

$$G(x, F, \operatorname{Adj} F, \det F) = W(x, F)$$

holds almost everywhere in Ω;

(b) coercivity: there exist constants $\alpha > 0$ and $r > 1$ as well as a function $g \in L_1(\Omega)$ such that

$$W(x,F) \geq \alpha(\|F\|^n + (\det F)^r) + g(x) \tag{4.1}$$

for almost all $x \in \Omega$ and all $F \in \mathbb{M}^n_{\geq 0}$.

Given $\overline{\varphi} : \Omega \to \Omega'$ with $\overline{\varphi} \in W^1_n(\Omega)$ and a measurable function $M : \Omega \to \mathbb{R}$, define the class of admissible deformations

$$\mathcal{A} = \{\varphi \in W^1_1(\Omega) \cap FD(\Omega),\ I(\varphi) < \infty,\ \frac{|D\varphi(x)|^n}{J(x,\varphi)} \leq M(x) \in L_s(\Omega),$$
$$s > n-1,\ \varphi|_\Gamma = \overline{\varphi}|_\Gamma \text{ a. e. in } \Gamma,\ J(x,\varphi) \geq 0 \text{ a. e. in } \Omega\}.$$

Remark 4.1. Here we understand $\psi|_\Gamma = \overline{\varphi}|_\Gamma$ in the sense of traces, that is, $\psi - \overline{\varphi} \in \overset{\circ}{W}{}^1_n(\Omega)$.

Theorem 4.1 (Existence theorem)**.** *Suppose that:*

1. *conditions* **(a)** *and* **(b)** *on the function $W(x,F)$ are fulfilled;*
2. *$\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ is a homeomorphism;*
3. *the set $\mathcal{A}$ is nonempty.*

Then there exists at least one mapping $\varphi_0 \in \mathcal{A}$ such that

$$I(\varphi_0) = \inf_{\varphi \in \mathcal{A}} I(\varphi).$$

Moreover, $\varphi_0 : \overline{\Omega} \to \overline{\Omega'}$ is a homeomorphism.

4.2 Proof of the existence theorem

We subdivide the proof of the existence theorem into three steps. On the first step (see subsection 4.2.1) we establish that the weak limit of a minimizing sequence exists. On the second step we investigate the main properties of the mappings of class $\mathcal{A}$ (subsection 4.2.2) and the limit mapping φ_0 (subsections 4.2.3–4.2.7), as well as verify that φ_0 belongs to the class $\mathcal{A}$ of admissible deformations. This is a key step since we introduce a new class of admissible deformations, and consequently, the verification of containment in it

differs substantially from previous works. Our proof uses both classical theorems of functional analysis and properties of mappings with finite distortion obtained quite recently [33]. Finally, on the third step it remains to show that the mapping found is actually a solution to the minimization problem, which requires proving that the energy functional is lower semicontinuous (subsection 4.2.8).

4.2.1 Existence of a minimizing mapping.

Let us prove the existence of a minimizing mapping for the functional

$$\overline{I}(\varphi) = I(\varphi) - \int\limits_{\Omega} g(x)\,dx.$$

Lemma 4.1. *If Ω is a domain in $\mathbb{R}^n$ with locally Lipschitz boundary and φ_0, $\varphi_k \in W^1_n(\Omega)$ then $M^p_q(D\varphi_0) \in L_{\frac{n}{m}}(\Omega)$, where $p = (p_1, p_2, \dots, p_m)$ and $q = (q_1, q_2, \dots, q_m)$, while $M^p_q(D\varphi_0)$ is the minor of the matrix $D\varphi_0$ of size m, with $1 \le m \le n$, consisting of the entries $\dfrac{\partial(\varphi_0)_{p_i}}{\partial x_{q_j}}$ (that is, of the rows $p_1, p_2, \dots p_m$ and columns $q_1, q_2, \dots q_m$), and if $\varphi_k \to \varphi_0$ weakly in $W^1_n(\Omega)$, while $M^p_q(D\varphi_k) \to H^p_q$ weakly in $L_{\frac{n}{m}}(\Omega)$ for all m with $1 \le m \le n$, then $H^p_q = M^p_q(D\varphi)$ for all m.*

Proof. Since $D\varphi_0 \in L_n(\Omega)$, applying Hölder's inequality and the boundedness of Ω, we easily verify that $M^p_q(D\varphi_0) \in L_{\frac{n}{m}}(\Omega)$.

Further we prove by induction on m. In the case $m = 1$ the assertion follows directly from the definition of φ_k, $\varphi_0 \in W^1_n(\Omega)$.

For sufficiently smooth functions ($C^n(\Omega)$) expanding the determinant along the first row yields

$$\begin{aligned} M^p_q(Df) &= \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial f_{p_1}}{\partial x_{q_j}} M^{\hat{p}_1}_{\hat{q}_j} = \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_{q_j}} \Big(f_{p_1} M^{\hat{p}_1}_{\hat{q}_j} \Big) \\ &\quad + f_{p_1} \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_{q_j}} M^{\hat{p}_1}_{\hat{q}_j} = \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial}{\partial x_{q_j}} \Big(f_{p_1} M^{\hat{p}_1}_{\hat{q}_j} \Big), \end{aligned}$$

where $\hat{q}_j = (q_1, q_2, \dots q_{j-1}, q_{j+1}, \dots q_m)$. The second term in the right-hand side vanishes not only for smooth functions (which can be verified directly), but also for the functions of class $L_{\frac{n}{m}}(\Omega)$ (Lemma 3.1).

Given a test function $\theta \in \mathcal{D}(\Omega)$, for $\xi \in W^1_n(\Omega)$ and $\eta \in L_{\frac{n}{m-1}}(\Omega)$ the bilinear mapping

$$(\xi, \eta) \mapsto \int\limits_{\Omega} \xi M^{\hat{p}_1}_{\hat{q}_j}(D\eta) \frac{\partial \theta}{\partial x_{q_j}} \, dx$$

is continuous by Hölder's inequality. Indeed,

$$\begin{aligned}\left| \int\limits_{\Omega} \xi M^{\hat{p}_1}_{\hat{q}_j}(D\eta) \frac{\partial \theta}{\partial x_{q_j}} \, dx \right| &\leq \|\xi\|_n \|M^{\hat{p}_1}_{\hat{q}_j}(D\eta)\|_{\frac{n}{m}} \left\| \frac{\partial \theta}{\partial x_{q_j}} \right\|_{\frac{n-1}{n-m-1}} \\ &\leq C \|\xi\|_n \|M^{\hat{p}_1}_{\hat{q}_j}(D\eta)\|_{\frac{n}{m}}.\end{aligned}$$

Since the embedding of $W^1_n(\Omega)$ into $L_p(\Omega)$ is compact, extracting a subsequence if necessary, we may assume that the sequence φ_k converges strongly in $L_p(\Omega)$, while the sequence $M^{\hat{p}_1}_{\hat{q}_j}(D\eta)$ converges weakly in $L_{\frac{n}{m-1}}(\Omega)$ by the inductive assumption. Consequently, Theorem 3.4 yields the convergence

$$\int\limits_{\Omega} (\varphi_k)_{p_1} M^{\hat{p}_1}_{\hat{q}_j}(D\varphi_k) \frac{\partial \theta}{\partial x_{q_j}} \, dx \xrightarrow[k\to\infty]{} \int\limits_{\Omega} (\varphi_0)_{p_1} M^{\hat{p}_1}_{\hat{q}_j}(D\varphi_0) \frac{\partial \theta}{\partial x_{q_j}} \, dx.$$

Therefore,

$$\begin{aligned}\int\limits_{\Omega} M^p_q(D\varphi_k)\theta \, dx &= \sum_{j=1}^{n} (-1)^{j-1} \int\limits_{\Omega} \frac{\partial}{\partial x_{q_j}} \Big((\varphi_k)_{p_1} M^{\hat{p}_1}_{\hat{q}_j}(D\varphi_k) \Big) \theta \, dx \\ &= \sum_{j=1}^{n} (-1)^{j} \int\limits_{\Omega} (\varphi_k)_{p_1} M^{\hat{p}_1}_{\hat{q}_j}(D\varphi_k) \frac{\partial \theta}{\partial x_{q_j}} \, dx \\ &\xrightarrow[k\to\infty]{} \sum_{j=1}^{n} (-1)^{j} \int\limits_{\Omega} (\varphi_0)_{p_1} M^{\hat{p}_1}_{\hat{q}_j}(D\varphi_0) \frac{\partial \theta}{\partial x_{q_j}} \, dx \\ &= \int\limits_{\Omega} M^p_q(D\varphi_0)\theta \, dx,\end{aligned}$$

which completes the proof of the lemma. □

Observe now that the coercivity (4.1) of the function W, Lemma 4.1, and Poincaré inequality (Theorem 3.3) ensure the existence of constants $c > 0$

and $d \in \mathbb{R}$ such that

$$\overline{I}(\varphi) = I(\varphi) - \int\limits_{\Omega} g(x)\,dx \geq c\Big(\|\varphi \mid W_n^1(\Omega)\|^n + \|\operatorname{Adj} D\varphi \mid L_{\frac{n}{n-1}}(\Omega)\|^{\frac{n}{n-1}} + \|J(\cdot,\varphi) \mid L_r(\Omega)\|^r \Big) + d \quad (4.2)$$

for every mapping $\varphi \in \mathcal{A}$.

Take a minimizing sequence $\{\varphi_k\}$ for the functional $\overline{I}$. Then

$$\lim_{k\to\infty} \overline{I}(\varphi_k) = \inf_{\varphi\in\mathcal{A}} \overline{I}(\varphi).$$

By (4.2) and the assumption $\inf\limits_{\varphi\in\mathcal{A}} \overline{I}(\varphi) < \infty$ we conclude that the sequence $(\varphi_k,\ \operatorname{Adj} D\varphi_k,\ J(\varphi_k))$ is bounded in the reflexive Banach space $W_n^1(\Omega) \times L_{\frac{n}{n-1}}(\Omega) \times L_r(\Omega)$. Consequently, there exists a subsequence (which we also denote by $\{(\varphi_k,\ \operatorname{Adj} D\varphi_k,\ J(\varphi_k))\}_{k\in\mathbb{N}}$) weakly converging to an element $(\varphi_0, H, \delta) \in W_n^1(\Omega) \times L_{\frac{n}{n-1}}(\Omega) \times L_r(\Omega)$, and furthermore, $H = \operatorname{Adj} D\varphi_0$ and $\delta = J(\cdot,\varphi_0)$ by Lemma 4.1. Hence, there exists a minimizing sequence satisfying the conditions

$$\begin{cases} \varphi_k \longrightarrow \varphi_0 & \text{weakly in } W_n^1(\Omega), \\ \operatorname{Adj} D\varphi_k \longrightarrow \operatorname{Adj} D\varphi_0 & \text{weakly in } L_{\frac{n}{n-1}}(\Omega), \\ J(\cdot,\varphi_k) \longrightarrow J(\cdot,\varphi_0) & \text{weakly in } L_r(\Omega) \end{cases} \quad (4.3)$$

as $k \to \infty$, where φ_0 guarantees the sharp lower bound $\overline{I}(\varphi_0) = \inf\limits_{\varphi\in\mathcal{A}} \overline{I}(\varphi)$. It remains to verify that $\varphi_0 \in \mathcal{A}$. To this end, we need the properties of mappings of class $\mathcal{A}$ presented in the next subsection.

4.2.2 Properties of admissible deformations $\varphi \in \mathcal{A}$.

Let us state some properties of mappings of class $\mathcal{A}$.

Remark 4.2. If $\varphi \in \mathcal{A}$ is a homeomorphism then by Theorem 3.9 it induces a bounded composition operator $\varphi^* : L_n^1(\Omega') \to L_q^1(\Omega)$, where $q = \frac{ns}{s+1}$ and $\varphi^*(f) = f \circ \varphi$. Furthermore, we have the estimate

$$\|\varphi^*\| \leq \|K_{\varphi,n}(\cdot) \mid L_{ns}(\Omega)\| \leq C\|\varphi^*\| \quad (4.4)$$

with some constant C.

Remark 4.3. If $\varphi \in \mathcal{A}$ is a homeomorphism then by Theorem 3.10 and remark 4.2 the inverse mapping $\psi = \varphi^{-1}$ induces a bounded operator $\psi^* : L^1_{q'}(\Omega) \to L^1_n(\Omega')$, where $q' = \frac{ns}{s-n+1}$, and

$$\|\psi^*\| \leq \|K_{\psi,q'}(\cdot) \mid L_\varrho(\Omega')\| \leq \|K_{\varphi,n}(\cdot) \mid L_{ns}(\Omega)\|^{n-1}, \tag{4.5}$$

where $\frac{1}{\varrho} = \frac{1}{n} - \frac{1}{q'} = \frac{n-1}{ns}$.

Lemma 4.2. *If $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ is a homeomorphism then so is $\varphi : \overline{\Omega} \to \overline{\Omega'}$ with $\varphi|_\Omega \in \mathcal{A}$.*

Proof. The domain Ω (Ω') has locally Lipschitz boundary, that is, there exists a tuple of charts $\{\nu_j,\, U_j\}$ ($\{\mu_k,\, V_k\}$), where

$$\nu_j : U_j \to B(0, r_j) \subset \mathbb{R}^n \quad (\{\mu_k : V_k \to B'(0, r_k) \subset \mathbb{R}^n\})$$

and

$$\begin{aligned} \nu_j(U_j \cap \Omega) = B(0, r_j) \cap \{x_n > 0\} = O_j \text{ and} \\ \nu_j(U_j \cap \Gamma) = B(0, r_j) \cap \{x_n = 0\} \\ (\mu_j(V_j \cap \Omega') = B'(0, r_k) \cap \{y_n > 0\} = O'_k \text{ and} \\ \mu_k(V_k \cap \Gamma') = B'(0, r_k) \cap \{y_n = 0\}), \end{aligned}$$

each mapping ν_k (μ_k) is a quasi-isometric mapping.

Then the mapping $\mu_k \circ \varphi \circ \nu_j^{-1} : O_{jk} \to O'_{jk}$, where $O_{jk} = \nu_j \circ \varphi^{-1}(\mu_k^{-1}(O'_k))$ and $O'_{kj} = \mu_k \circ \varphi(\nu_j^{-1}(Q_j))$, is of Sobolev class $W^1_n(O_{jk})$ (see [24, Theorem 1] for instance).

Consider the symmetrization $\varphi_{jk,sym} : O_{jk} \cup \widetilde{O}_{jk} \to O'_{kj} \cup \widetilde{O}'_{kj}$ of the mapping $\mu_k \circ \varphi \circ \nu_j^{-1}$, where $\widetilde{O}_{jk} = \{(x_1, x_2, \dots, -x_n) \mid (x_1, x_2, \dots, x_n) \in O_{jk}\}$ is the reflection of the set O_{jk} in the hyperplane $Ox_1x_2 \dots x_{n-1}$ and $\widetilde{O}'_{kj} = \{(y_1, y_2, \dots, -y_n) \mid (y_1, y_2, \dots, y_n) \in O'\}$ is the reflection of the set O'_{kj} in the hyperplane $O'y_1y_2 \dots y_{n-1}$,

$$\varphi_{jk,sym}(x_1, x_2, \dots, x_n) = \begin{cases} \mu_k \circ \varphi \circ \nu_j^{-1}(x_1, x_2, \dots, x_n) & \text{if } x_n \geq 0, \\ \iota_n \circ \mu_k \circ \varphi \circ \nu_j^{-1}(x_1, x_2, \dots, -x_n) & \text{if } x_n < 0, \end{cases}$$

where the mapping $\iota_n : \mathbb{R}^n \to \mathbb{R}^n$ acts as $\iota_n(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, -x_n)$.

Since $\varphi_{jk,sym} \in W_n^1(O_{jk} \cup \widetilde{O}_{jk})$, by Theorem 3.7 this mapping is either continuous, open and discrete, or constant. In our case it is not a constant.

Furthermore, we have the coincidence of mappings:

$$\varphi_{jk,sym}|_{U_j \cap \varphi^{-1}(V_k) \cap \Gamma} = \overline{\varphi}|_{U_j \cap \varphi^{-1}(V_k) \cap \Gamma}.$$

Further, observe that φ and $\overline{\varphi}$ are homotopic mappings since they coincide on the boundary. Consequently, the degree $\mu(\varphi, \Omega)$ of φ equals the degree $\mu(\overline{\varphi}, \Omega)$ of $\overline{\varphi}$, and moreover, $\mu(\overline{\varphi}, \Omega) = 1$. Since φ is sense preserving by Lemma 3.3, each point $y \in \Omega'$ has exactly one preimage in Ω (for more details, see [25, Proposition 4.10]); therefore, φ is bijective. The inverse mapping φ^{-1} is continuous because φ is open (by Theorem 3.7). Thus, we conclude that φ is a homeomorphism. □

Lemma 4.3. *Consider a homeomorphism $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ and a sequence $\{\varphi_k\} \subset \mathcal{A}$ weakly converging to φ in $W_n^1(\Omega)$. If $\psi_k = \varphi_k^{-1}$ then there exists a subsequence $\{\psi_{k_l}\}$ such that $\psi_{k_l}(y) \to \psi_0(y)$ uniformly up to the boundary.*

Proof. Since each $\varphi_k : \Omega \to \Omega'$ is a homeomorphism by Lemma 4.2, it follows that all mappings $\psi_k = \varphi_k^{-1}$ are well-defined.

Since the domain Ω' is bounded and $\psi_k|_{\Gamma'} = \overline{\varphi}^{-1}|_{\Gamma'}$, the sequence ψ_k is uniformly bounded.

On the other hand, $f \in W_n^1(\mathbb{R}^n)$ satisfies the estimate (corollary of Lemma 4.1 of [20])

$$\operatorname{osc}(f,\ S(y',r)) \le L\biggl(\ln \frac{r_0}{r}\biggr)^{-\frac{1}{n}} \Biggl(\int\limits_{B(y',r_0)} |Df(y)|^n dy\Biggr)^{\frac{1}{n}},$$

where $S(y', r)$ is the sphere of radius $r < \dfrac{r_0}{2}$ centered at y' and $B(y', r_0)$ is the ball of radius r_0 centered at y'.

Thus, for $\psi_k \in W_n^1(\Omega')$ we obtain

$$\operatorname{osc}(\psi_k,\ S(y',r)) \le L\biggl(\ln \frac{r_0}{r}\biggr)^{-\frac{1}{n}} \Biggl(\int\limits_{B(y',r_0)} |D\psi_k(y)|^n dy\Biggr)^{\frac{1}{n}},$$

if $B(y', r_0) \subset \Omega'$. It follows the equicontinuity of the family of functions $\{\psi_k\}_{k\in\mathbb{N}}$ on any compact part of Ω'.

Indeed, Hölder's inequality and the estimate (4.5) yield

$$\begin{aligned}
\int\limits_{B(y',r_0)} |D\psi_k(y)|^n dy &\leq \int\limits_{B(y',r_0)} \frac{|D\psi_k(y)|^n}{J(y,\psi_k)^{\frac{n}{q'}}} J(y,\psi_k)^{\frac{n}{q'}}\, dy \\
&\leq \left(\int\limits_{\Omega'} \left(\frac{|D\psi_k(y)|^n}{J(y,\psi_k)^{\frac{n}{q'}}}\right)^{\frac{\varrho}{n}} dy\right)^{\frac{n}{\varrho}} \left(\int\limits_{B(y',r_0)} J(y,\psi_k)^{\frac{n}{q'}\cdot\frac{\varrho}{\varrho-n}}\, dy\right)^{\frac{\varrho-n}{\varrho}} \\
&\leq \|K_{\psi_k,q'}(\cdot) \mid L_{\varrho}(\Omega')\|^n |\psi_k(B(y',r_0))|^{\frac{s}{s-(n-1)}} \\
&\leq \|K_{\varphi_k,n}(\cdot) \mid L_{ns}(\Omega)\|^{n(n-1)} |\psi_k(B(y',r_0))|^{\frac{s}{s-(n-1)}} \\
&\leq \|M(\cdot) \mid L_s(\Omega)\|^{(n-1)} |\Omega|^{\frac{s}{s-(n-1)}}
\end{aligned}$$

since

$$\frac{n}{q'}\cdot\frac{\varrho}{\varrho-n} = 1 \quad \text{and} \quad \frac{\varrho-n}{\varrho} = \frac{s}{s-(n-1)}.$$

To show the equicontinuity of the family of functions $\{\psi_k\}_{k\in\mathbb{N}}$ near $\partial\Omega'$ we use the method introduced in papers [11, 27, 35]. We fix an arbitrary ball $B \Subset \Omega$ such that for all $k \geq k_0$, where k_0 is big enough, the inclusion $\varphi_k(B) \subset B' \Subset \Omega'$ holds (here B' is a ball). Consider two points $x, y \in \Omega' \setminus B'$ and an arbitrary curve $\gamma : [0,1] \to \Omega' \setminus B'$ with endpoints x and y.

Definition 13. The quantity

$$\operatorname{cap}(B',\gamma; L^1_n(\Omega')) = \inf_u \int\limits_{\Omega'} |Du|^n\, dy,$$

where the lower bound is taken over all continuous functions $u \in L^1_n(\Omega')$ such that $u = 0$ on B' and $u \geq 1$ on γ, is called *capacity* of the pair sets (B',γ).

If we now define

$$d'_n(x,y) = \inf_\gamma \operatorname{cap}(B',\gamma; L^1_n(\Omega'))$$

where the lower bound is taken over all curves $\gamma : [0,1] \to \Omega' \setminus B'$ with endpoints x and y, we obtain a metric on the set $\Omega' \setminus B'$ [11, 35].

By Remark 4.2 the composition operator $\varphi_k^* : L^1_n(\Omega') \to L^1_q(\Omega)$, where $q = \frac{ns}{s+1}$ and $\varphi_k^*(f) = f \circ \varphi_k$ is bounded and by Lemma 4.6 below $\|\varphi_k^*\| \leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{1}{n}}$.

From here and (4.4) we infer

$$\operatorname{cap}(B, \varphi_k^{-1}(\gamma); L_q^1(\Omega))^{\frac{1}{q}} \leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{1}{n}} \operatorname{cap}(B', \gamma; L_n^1(\Omega'))^{\frac{1}{n}}.$$

The last estimate implies

$$d_q(\psi_k(x), \psi_k(y)) \leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{q}{n}} d'_n(x,y)^{\frac{q}{n}}.$$

As soon as domain Ω' (Ω) meets (ϵ, δ)-condition in the sense of paper [15] there exists an extension operator $i' : L_n^1(\Omega') \to L_n^1(\mathbb{R}^n)$ $(i' : L_q^1(\Omega) \to L_q^1(\mathbb{R}^n))$. It follows that the topology of the metric space $(\Omega' \setminus B', d'_n)$ ($(\Omega \setminus B, d_q)$) is equivalent to the Euclidean one. The last property is a consequence of the following two assertions:

1. *given $\rho > 0$, there exist $\tau > 0$ such that $|x-y| < \rho$ provided $d_q(x,y) < \tau$ for points x, $y \in \Omega$ closed enough to $\partial\Omega$;*

2. *given $\tau > 0$, there exist $\varkappa > 0$ such that $d'_n(x,y) < \tau$ provided $|x-y| < \varkappa$ for points x, $y \in \Omega'$ closed enough to $\partial\Omega'$.*

Thus, we see that the family $\{\psi_k,\, k \in \mathbb{N}\}$ is equicontinuous and uniformly bounded. By the Arzelà–Ascoli theorem (see [16] for instance) there exists a subsequence $\{\psi_{k_l}\}$ converging uniformly to a mapping ψ_0 as $k_l \to \infty$. $\square$

4.2.3 Nonnegativity of the Jacobian.

Lemma 4.4. *The limit mapping φ_0 satisfies $J(\cdot, \varphi_0) \geq 0$ a. e. in Ω.*

Proof. The inequality $J(\cdot, \varphi_0) \geq 0$ follows directly from the weak convergence of $J(\cdot, \varphi_k)$ in $L_r(\Omega)$ with $r > 1$.

Even in the case $r = 1$ we can establish the nonnegativity of the Jacobian by using weak convergence (Theorem 3.6). Indeed, the sequence $\varphi_k \in W_n^1(\Omega)$ is bounded in $W_n^1(\Omega)$, and, since the embedding of $W_n^1(\Omega)$ into $L_1(\Omega)$ is compact (Theorem 3.1), there exists a subsequence converging in $L_1(\Omega)$. Hence, for every continuous function $f : \Omega \to \mathbb{R}$ with compact support in Ω we have

$$\int\limits_\Omega f(x) J(x, \varphi_k)\, dx \xrightarrow[k\to\infty]{} \int\limits_\Omega f(x) J(x, \varphi_0)\, dx.$$

It follows immediately

$$\int\limits_\Omega f(x) J(x, \varphi_0)\, dx \geq 0$$

for an arbitrary function $f(x) \geq 0$. Hence we imply $J(x, \varphi_0)\, dx \geq 0$ a. e. $\square$

4.2.4 Behavior on the boundary.

Lemma 4.5. *Equality $\varphi_0|_\Gamma = \overline{\varphi}|_\Gamma$ holds a. e. in Γ.*

Proof. Since the trace operator is compact, for every $1 < q < \infty$ we deduce

$$\varphi_k \longrightarrow \varphi_0 \text{ weakly in } W^1_n(\Omega) \;\Rightarrow\; \operatorname{tr}\varphi_k \longrightarrow \operatorname{tr}\varphi_0 \text{ in } L_p(\Gamma).$$

Extracting a subsequence converging almost everywhere in Γ, we obtain $\varphi_0|_\Gamma = \overline{\varphi}|_\Gamma$ almost everywhere in Γ. $\square$

4.2.5 Boundedness of the composition operator.

Lemma 4.6. *The mapping φ_0 induces a bounded composition operator $\varphi_0^* : L^1_n(\Omega') \cap \mathrm{Lip}(\Omega') \to L^1_q(\Omega)$.*

Proof. Consider $u \in L^1_n(\Omega') \cap \mathrm{Lip}(\Omega')$. Since (4.4) yields

$$\|\varphi_k^*\| \leq \|K_{\varphi_k,n}(\cdot) \mid L_{ns}(\Omega)\| \leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{1}{n}},$$

the sequence $w_k = \varphi_k^* u = u \circ \varphi_k$ is bounded in $L^1_q(\Omega)$. Using a compact embedding into the Sobolev space, we obtain a subsequence with $w_k \to w_0$ in $L_r(\Omega)$, where $r \leq \frac{ns}{ns-s-1}$. From this sequence, in turn, we can extract a subsequence converging almost everywhere in Ω. If $u \in L^1_n(\Omega') \cap \mathrm{Lip}(\Omega')$ then $w_0(x) = u \circ \varphi_0(x)$ for almost all $x \in \Omega$.

On the other hand, since w_k converges weakly to w_0 in $L^1_q(\Omega)$, we have

$$\begin{aligned}\|u \circ \varphi_0 \mid L^1_q(\Omega)\| = \|w_0 \mid L^1_q(\Omega)\| &\leq \varliminf_{k\to\infty} \|w_k \mid L^1_q(\Omega)\| \\ &= \varliminf_{k\to\infty} \|\varphi_k^*(u) \mid L^1_q(\Omega)\| \leq \varliminf_{k\to\infty} \|\varphi_k^*\| \cdot \|u \mid L^1_n(\Omega')\| \\ &\leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{1}{n}} \cdot \|u \mid L^1_n(\Omega')\|.\end{aligned}$$

Thus, φ_0 induces a bounded composition operator $\varphi_0^* : L^1_n(\Omega') \cap \mathrm{Lip}(\Omega') \to L^1_q(\Omega)$, and moreover, $\|\varphi_0^*\| \leq \|M(\cdot) \mid L_s(\Omega)\|^{\frac{1}{n}}$. $\square$

4.2.6 Injectivity.

Verify that the mapping $\varphi_0 : \Omega \to \overline{\Omega'}$ is injective almost everywhere (since φ_0 is the pointwise limit of the homeomorphisms $\varphi_k : \Omega \to \Omega'$, the images of some points $x \in \Omega$ may lie on the boundary $\partial\Omega'$). Recall the definition.

Definition 14. A mapping $\varphi : \Omega \to \overline{\Omega'}$ is called *injective almost everywhere* whenever there exists a negligible set S outside which φ is injective.

Denote by $S \subset \Omega$ a negligible set on which the convergence $\varphi_k(x) \to \varphi_0(x)$ as $k \to \infty$ fails. If $x \in \Omega \setminus S$ with $\varphi(x) \in \Omega'$ then the injectivity follows from the uniform convergence of ψ_k on Ω' (see Lemma 4.3) and the equality

$$\psi_k \circ \varphi_k(x) = x.$$

Passing to the limit as $k \to \infty$, we infer that

$$\psi_0 \circ \varphi_0(x) = x,\ x \in \Omega \setminus S.$$

Hence, we deduce that if $\varphi_0(x_1) = \varphi_0(x_2) \in \Omega'$ for two points $x_1, x_2 \in \Omega \setminus S$ then $x_1 = x_2$.

It remains to verify that the set of points $x \in \Omega$ with $\varphi(x) \in \partial\Omega'$ is negligible. The argument rests on the method of proof of [38, Theorem 4]. For the reader's convenience, we present here the new details of this method.

Given a bounded open set $A' \subset \mathbb{R}^n$, define the class of functions $\overset{\circ}{L}{}^1_p(A')$ as the closure of the subspace $C_0^\infty(A')$ in the seminorm of $L^1_p(A')$. In general, a function $f \in \overset{\circ}{L}{}^1_p(A')$ is defined only on the set A', but, extending it by zero, we may assume that $f \in L^1_p(\mathbb{R}^n)$.

Lemma 4.7 (cf. Lemma 1 of [38])**.** *Assume that the mapping $\varphi : \Omega \to \overline{\Omega'}$ induces a bounded composition operator*

$$\varphi^* : L^1_p(\Omega') \cap \operatorname{Lip}(\Omega') \to L^1_q(\Omega), \quad 1 \le q < p \le \infty.$$

Then

$$\Phi(A') = \sup_{f \in \overset{\circ}{L}{}^1_p(A') \cap \operatorname{Lip}(A')} \left(\frac{\|\varphi^* f \mid L^1_q(\Omega)\|}{\|f \mid L^1_p(A' \cap \Omega')\|} \right)^{\sigma}, \quad \sigma = \begin{cases} \frac{pq}{p-q} & \text{for } p < \infty, \\ q & \text{for } p = \infty, \end{cases}$$

is a bounded monotone countably additive function defined on the open bounded sets A' with $A' \cap \Omega' \neq \emptyset$.

Proof. It is obvious that $\Phi(A'_1) \le \Phi(A'_2)$ whenever $A'_1 \subset A'_2$.

Take disjoint sets A'_i, $i \in \mathbb{N}$ in Ω' and put $A'_0 = \bigcup\limits_{i=1}^{\infty} A'_i$. Consider a function $f_i \in \overset{\circ}{L}{}^1_p(A'_i) \cap \operatorname{Lip}(A'_i)$ such that the conditions

$$\|\varphi^* f_i \mid L^1_q(\Omega)\| \ge (\Phi(A'_i)(1 - \frac{\varepsilon}{2^i}))^{1/\sigma} \|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\|$$

and

$$\|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\|^p = \Phi(A'_i)(1 - \frac{\varepsilon}{2^i}) \text{ for } p < \infty$$
$$(\|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\|^p = 1 \text{ for } p = \infty)$$

hold simultaneously, where $0 < \varepsilon < 1$. Putting $f_N = \sum\limits_{i=1}^{N} f_i \in L^1_p(\Omega') \cap \operatorname{Lip}(\Omega')$, and applying Hölder's inequality (the case of equality), we obtain

$$\begin{aligned}
\|\varphi^* f_N \mid L^1_q(\Omega)\| &\ge \left(\sum_{i=1}^{N} \left(\Phi(A'_i) \Big(1 - \frac{\varepsilon}{2^i}\Big) \right)^{\frac{q}{\sigma}} \big\| f_i | \overset{\circ}{L}{}^1_p(A'_i) \big\|^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{N} \Phi(A'_i) \Big(1 - \frac{\varepsilon}{2^i}\Big) \right)^{\frac{1}{\sigma}} \left\| f_N \mid \overset{\circ}{L}{}^1_p\Big(\bigcup_{i=1}^{N} A'_i \Big) \right\| \\
&\ge \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon \Phi(A'_0) \right)^{\frac{1}{\sigma}} \left\| f_N \mid \overset{\circ}{L}{}^1_p\Big(\bigcup_{i=1}^{N} A'_i \Big) \right\|
\end{aligned}$$

since the set A_i, on which the functions $\nabla \varphi^* f_i$ are nonvanishing, are disjoint. This implies that

$$\Phi(A'_0)^{\frac{1}{\sigma}} \ge \sup \frac{\|\varphi^* f_N \mid L^1_p(\Omega)\|}{\left\| f_N \mid \overset{\circ}{L}{}^1_p\Big(\bigcup\limits_{i=1}^{N} A'_i \Big) \right\|} \ge \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon \Phi(A'_0) \right)^{\frac{1}{\sigma}},$$

where we take the sharp upper bound over all functions

$$f_N \in \overset{\circ}{L}{}^1_p\Big(\bigcup_{i=1}^{N} A'_i \Big) \cap \operatorname{Lip}\Big(\bigcup_{i=1}^{N} A'_i \Big), \quad f_N = \sum_{i=1}^{N} f_i,$$

and f_i are of the form indicated above. Since N and ε are arbitrary,

$$\sum_{i=1}^{\infty} \Phi(A_i') \leq \Phi\Big(\bigcup_{i=1}^{\infty} A_i'\Big).$$

We can verify the inverse inequality directly by using the definition of Φ. $\square$

Lemma 4.8. *Take a monotone countably additive function Φ defined on the bounded open sets A' with $A' \cap \Omega' \neq \emptyset$. For every set A' there exists a sequence of balls $\{B_j\}$ such that*

1. *the families of $\{B_j\}$ and $\{2B_j\}$ constitute finite coverings of U;*
2. $\sum\limits_{j=1}^{\infty} \Phi(2B_j) \leq \zeta_n \Phi(U)$, *where the constant ζ_n depends only on the dimension n.*

Proof. In accordance with Lemma 3.2 construct two sequences $\{B_j\}$ and $\{2B_j\}$ of balls and subdivide the latter into ζ_n subfamilies $\{2B_{1j}\}_{j=1}^{\infty}, \dots, \{2B_{\zeta_n j}\}_{j=1}^{\infty}$ so that in each tuple the balls are disjoint: $2B_{ki} \cap 2B_{kj} = \emptyset$ for $i \neq j$ and $k = 1, \dots, \zeta_n$. Consequently,

$$\sum_{j=1}^{\infty} \Phi(2B_j) = \sum_{k=1}^{\zeta_n} \sum_{j=1}^{\infty} \Phi(2B_{kj}) \leq \sum_{k=1}^{\zeta_n} \Phi(U) = \zeta_n \Phi(U).$$

$\square$

Lemma 4.9. *If a measurable almost everywhere injective mapping $\varphi : \Omega \to \overline{\Omega'}$ induces a bounded composition operator*

$$\varphi^* : L_p^1(\Omega') \cap \mathrm{Lip}(\Omega') \to L_q^1(\Omega), \quad 1 \leq q < p \leq n,$$

then $|\varphi^{-1}(E)| = 0$ for every set $E \subset \Gamma' = \partial\Omega'$.

Proof. Consider the cutoff $\eta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0,1)$ and vanishing outside $B(0,2)$. By Lemma 4.7 the function $f(y) = \eta\big(\frac{y-y_0}{r}\big)$ satisfies

$$\|\varphi^* f \mid L_q^1(\Omega)\| \leq C_1 \Phi(2B)^{\frac{1}{\sigma}} |B|^{\frac{1}{p}-\frac{1}{n}},$$

where $B \cap \Omega' \neq \emptyset$. Take an set $E \subset \Gamma'$ with $|E| = 0$. Since φ is a mapping with finite distortion, $\varphi^{-1}(E) \neq \Omega$ (otherwise, $J(x, \varphi) = 0$ and, consequently, $D\varphi(x) = 0$, that is, φ is a constant mapping). Hence, there is a cube $Q \subset \Omega$ such that $2Q \subset \Omega$ and $|Q \setminus \varphi^{-1}(E)| > 0$ (here $2Q$ is a cube with the same center as Q and the edges stretched by a factor of two compared to Q). Since φ is a measurable mapping, by Luzin's theorem there is a compact set $T \subset Q \setminus \varphi^{-1}(E)$ of positive measure such that $\varphi : T \to \Omega'$ is continuous. Then, the image $\varphi(T) \subset \Omega'$ is compact and $\varphi(T) \cap E = \emptyset$. Consider an open set $U \supset E$ with $\varphi(T) \cap U = \emptyset$ and $U \cap \Omega' \neq \emptyset$. Choose a tuple $\{B(y_i, r_i)\}$ of balls in accordance with Lemma 3.2: $\{B(y_i, r_i)\}$ and $\{B(y_i, 2r_i)\}$ are coverings of U, and the multiplicity of the covering $\{B(y_i, 2r_i)\}$ is finite ($B(y_i, 2r_i) \subset U$ for all $i \in \mathbb{N}$). Then the function f_i associated to the ball $B(y_i, r_i)$ satisfies $\varphi^* f_i = 1$ on $\varphi^{-1}(B(y_i, r_i))$ and $\varphi^* f = 0$ outside $\varphi^{-1}(B(y_i, 2r_i))$, In particular, $\varphi^* f_i = 0$ on T. In addition, we have the estimate

$$\|\varphi^* f_i \mid L^1_q(2Q)\| \leq \|\varphi^* f_i \mid L^1_q(\Omega)\| \leq C_1 \Phi(B(y_i, 2r_i))^{\frac{1}{\sigma}} |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{n}}.$$

By Poincaré inequality (see [18] for instance), for every function $g \in W^1_{q,\mathrm{loc}}(Q)$, where $q < n$, vanishing on T we have

$$\biggl(\int\limits_Q |g|^{q^*}\, dx\biggr)^{1/q^*} \leq C_2 l(Q)^{n/q^*} \biggl(\int\limits_{2Q} |\nabla g|^q\, dx\biggr)^{1/q},$$

where $q^* = \frac{nq}{n-q}$ and $l(Q)$ is the edge length of Q.

Applying Poincaré inequality to the function $\varphi^* f_i$ and using the last two estimates, we obtain

$$|\varphi^{-1}(B(y_i, r_i)) \cap Q|^{\frac{1}{q} - \frac{1}{n}} \leq C_3 \Phi(B(y_i, 2r_i))^{\frac{1}{\sigma}} |B(y_i, r_i)|^{\frac{1}{p} - \frac{1}{n}}.$$

In turn, Hölder's inequality guarantees that

$$\biggl(\sum_{i=1}^{\infty} |\varphi^{-1}(B(y_i, r_i)) \cap Q|\biggr)^{\frac{1}{q} - \frac{1}{n}} \\ \leq C_3 \biggl(\sum_{i=1}^{\infty} \Phi(B(y_i, 2r_i))\biggr)^{\frac{1}{\sigma}} \biggl(\sum_{i=1}^{\infty} |B(y_i, r_i)|\biggr)^{\frac{1}{p} - \frac{1}{n}}.$$

As the open set U is arbitrary, this estimate yields $|\varphi^{-1}(E) \cap Q| = 0$. Since the cube $Q \subset \Omega$ is arbitrary, it follows that $|\varphi^{-1}(E)| = 0$. $\square$

Since for the domain Ω' with locally Lipschitz boundary we have $|\partial\Omega'| = 0$, Lemmas 4.6 and 4.9 implies the next lemma.

Lemma 4.10. *Consider a mapping $\varphi_0 : \Omega \to \Omega'$, where Ω, $\Omega' \subset \mathbb{R}^n$ are domains with locally Lipschitz boundary, with $\varphi_0 \in W^1_n(\Omega)$ and $J(x, \varphi_0) \geq 0$ a. e. in Ω, such that*

1. *there exists a sequence of homeomorphisms $\varphi_k \in W^1_n(\Omega) \cap FD(\Omega)$ with $J(x, \varphi_k) \geq 0$ almost everywhere in Ω, such that $\varphi_k \to \varphi_0$ weakly in $W^1_n(\Omega)$;*
2. *each mapping φ_k induces a bounded composition operator $\varphi_k^* : L^1_n(\Omega') \to L^1_q(\Omega)$ with $q = \frac{ns}{s+1}$, where $\varphi_k^*(f) = f \circ \varphi_k$;*
3. *the norms of the operators $\|\varphi_k^*\|$ are jointly bounded;*
4. *$\varphi_k|_{\partial\Omega} = \varphi_0|_{\partial\Omega}$.*

Then the mapping φ_0 is injective almost everywhere.

Let us mention another interesting corollary of Theorem 3.8.

Lemma 4.11. *If an almost everywhere injective mapping $\varphi : \Omega \to \Omega'$ with $\varphi \in W^1_n(\Omega)$ and $J(x, \varphi) \geq 0$ a. e. in Ω has Luzin $\mathcal{N}^{-1}$-property then $J(x, \varphi) > 0$ for almost all $x \in \Omega$.*

Proof. Denote by E a set outside which the mapping φ is approximatively differentiable and has Luzin $\mathcal{N}^{-1}$-property. Since $\varphi \in W^1_n(\Omega)$, it follows that $|E| = 0$ (see [39, 13]). In addition, we may assume that

$$Z = \{x \in \Omega \setminus E \mid J(x, \varphi) = 0\}$$

is a Borel set. Put $\sigma = \varphi(Z)$. By the change-of-variable formula (Theorem 3.11), taking the injectivity of φ into account, we obtain

$$\int\limits_{\Omega\setminus\Sigma} \chi_Z(x) J(x, \varphi)\, dx = \int\limits_{\Omega\setminus\Sigma} (\chi_\sigma \circ \varphi)(x) J(x, \varphi)\, dx = \int\limits_{\Omega'} \chi_\sigma(y)\, dy.$$

By construction, the expression in the left-hand side vanishes; consequently, $|\sigma| = 0$. On the other hand, since φ has Luzin $\mathcal{N}^{-1}$-property, we have $|Z| = 0$. □

Using Lemma 4.11, we conclude that the limit mapping φ_0 satisfies the strict inequality $J(x, \varphi_0) > 0$ a. e. in Ω.

4.2.7 Behavior of the distortion coefficient.

The next lemma follows directly from [38, Theorem 1].

Lemma 4.12. *If an almost everywhere injective mapping $\varphi : \Omega \to \Omega'$ generates a bounded composition operator $\varphi^* : L^1_n(\Omega') \cap \mathrm{Lip}(\Omega') \to L^1_q(\Omega)$ with $q = \frac{ns}{s+1}$ then*

1. *$\varphi \in ACL(\Omega)$;*
2. *φ has finite distortion;*
3. *$\|K_{\varphi,n}(\cdot) \mid L_{ns}(\Omega)\| \leq \widetilde{M} < \infty$.*

Indeed, for each almost everywhere injective mapping φ the function $H_q(y)$ of [38] becomes quite simple:

$$H_q(y) = \begin{cases} \dfrac{|D\varphi(x)|}{|J(x,\varphi)|^{1/q}} & \text{if } y = \varphi(x),\ x \in \Omega \setminus (Z \cup \Sigma \cup I), \\ 0 & \text{otherwise.} \end{cases}$$

The necessary relations follow from the change-of-variable formula and the inequality $J(x,\varphi) \geq 0$:

$$\begin{aligned} H_{p,q}(\Omega')^{\varkappa} = \|H_q(\cdot) \mid L_{\varkappa}(\Omega')\|^{\varkappa} = \int\limits_{\Omega'} (H_q(y))^{\frac{pq}{p-q}}\, dy \\ = \int\limits_{\Omega\setminus(Z\cup\Sigma\cup I)} \left(\frac{|D\varphi(x)|}{|J(x,\varphi)|^{1/q}}\right)^{\frac{p}{p-q}} J(x,\varphi)\, dx = \int\limits_{\Omega\setminus(Z\cup\Sigma\cup I)} \frac{|D\varphi(x)|^{\frac{pq}{p-q}}}{|J(x,\varphi)|^{\frac{q}{p-q}}}\, dx \\ = \int\limits_{\Omega\setminus(Z\cup\Sigma\cup I)} \frac{|D\varphi(x)|^{ns}}{|J(x,\varphi)|^{s}}\, dx = \|K_{\varphi,n}(\cdot) \mid L_{ns}(\Omega)\|^{ns}. \end{aligned}$$

By Theorem 3.7, the mapping $\varphi_0 \in W^1_n(\Omega)$ is continuous, open, and discrete; furthermore, $\varphi_0|_\Gamma = \overline{\varphi}|_\Gamma$, and so φ_0 is a homeomorphism by Lemma 4.2. It remains to verify that the pointwise inequality

$$\frac{|D\varphi_0(x)|^n}{J(x,\varphi_0)} \leq M(x)$$

holds almost everywhere in Ω.

Lemma 4.13. *Consider a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ of mappings $\varphi_k : \Omega \to \Omega'$ with finite distortion weakly converging in $W^1_n(\Omega)$ to a mapping $\varphi_0 : \Omega \to \Omega'$. Assume that $J(x,\varphi_k) \geq 0$ almost everywhere in Ω. Suppose also that there exists an almost everywhere nonnegative function $M(x) \in L_s(\Omega)$ such that*

$$K_{\varphi_k,n}(x) \leq M(x)^{\frac{1}{n}} \text{ for all } k \in \mathbb{N}, \text{ for almost all } x \in \Omega.$$

Then the limit mapping φ satisfies

$$K_{\varphi_0,n}(x) \leq M(x)^{\frac{1}{n}} \text{ for almost all } x \in \Omega.$$

Proof. Take a test function $\theta \in C_0^\infty(\Omega)$. Hölder's inequality yields

$$\begin{aligned}
\int\limits_\Omega |D\varphi_j(x)|^{\frac{ns}{s+1}}\theta(x)\,dx &= \int\limits_\Omega \frac{|D\varphi_j(x)|^{\frac{ns}{s+1}}}{J(x,\varphi_j)^{\frac{s}{s+1}}} J(x,\varphi_j)^{\frac{s}{s+1}}\theta(x)\,dx \\
&\leq \Bigg(\int\limits_\Omega \frac{|D\varphi_j(x)|^{\frac{ns}{s+1}(s+1)}}{J(x,\varphi_j)^{\frac{s}{s+1}(s+1)}}\theta(x)^{\frac{1}{s+1}(s+1)}\,dx\Bigg)^{\frac{1}{s+1}} \\
&\quad\times \Bigg(\int\limits_\Omega J(x,\varphi_j)^{\frac{s}{s+1}\frac{s+1}{s}}\theta(x)^{\frac{s}{s+1}\frac{s+1}{s}}\,dx\Bigg)^{\frac{s}{s+1}} \\
&\leq \Bigg(\int\limits_\Omega K^{ns}_{\varphi_j,n}(x)\theta(x)\,dx\Bigg)^{\frac{1}{s+1}} \Bigg(\int\limits_\Omega J(x,\varphi_j)\theta(x)\,dx\Bigg)^{\frac{s}{s+1}} \\
&\leq \Bigg(\int\limits_\Omega M^s(x)\theta(x)\,dx\Bigg)^{\frac{1}{s+1}} \Bigg(\int\limits_\Omega J(x,\varphi_j)\theta(x)\,dx\Bigg)^{\frac{s}{s+1}}.
\end{aligned}$$

By lower semicontinuity, [23, Ch. 3, § 3], we can estimate the left-hand side:

$$\int\limits_\Omega |D\varphi_0(x)|^{\frac{ns}{s+1}}\theta(x)\,dx \leq \varliminf_{j\to\infty} \int\limits_\Omega |D\varphi_j(x)|^{\frac{ns}{s+1}}\theta(x)\,dx.$$

On the other hand, Theorem 3.6 yields

$$\int\limits_\Omega J(x,\varphi_j)\theta(x)\,dx \xrightarrow[j\to\infty]{} \int\limits_\Omega J(x,\varphi_0)\theta(x)\,dx$$

for every function $\theta \in C_0(\Omega)$. Finally, we obtain the inequality

$$\int\limits_{\Omega} |D\varphi_0(x)|^{\frac{ns}{s+1}}\theta(x)\,dx \\ \leq \left(\int\limits_{\Omega} M^s(x)\theta(x)\,dx\right)^{\frac{1}{s+1}} \left(\int\limits_{\Omega} J(x,\varphi_0)\theta(x)\,dx\right)^{\frac{s}{s+1}}. \quad (4.6)$$

Consider the family of functions $\theta_{r,\varepsilon,y} \in C_0^\infty(\Omega)$:

$$\theta_{r,\varepsilon,x_0}(x) = \begin{cases} 1 & \text{if } x \in B(x_0, r), \\ 0 & \text{if } x \notin B(x_0, r+\varepsilon), \\ 0 < \theta_{r,\varepsilon,x_0}(x) < 1 & \text{otherwise.} \end{cases}$$

And insert these functions into (4.6) in place of $\theta(x)$. Passing to the limit as $\varepsilon \to 0$, we obtain

$$\int\limits_{B(y,r)} |D\varphi_0(x)|^{\frac{ns}{s+1}}\,dx = \lim_{\varepsilon\to 0}\int\limits_{\Omega} |D\varphi_0(x)|^{\frac{ns}{s+1}}\theta_{r,\varepsilon,y}(x)\,dx \\ \leq \left(\lim_{\varepsilon\to 0}\int\limits_{\Omega} M^s(x)\theta_{r,\varepsilon,y}(x)\,dx\right)^{\frac{1}{s+1}} \left(\lim_{\varepsilon\to 0}\int\limits_{\Omega} J(x,\varphi_0)\theta_{r,\varepsilon,y}(x)\,dx\right)^{\frac{s}{s+1}} \\ = \left(\int\limits_{B(y,r)} M^s(x)\,dx\right)^{\frac{1}{s+1}} \left(\int\limits_{B(y,r)} J(x,\varphi_0)\,dx\right)^{\frac{s}{s+1}}.$$

Then, divide by the measure of the ball and pass to the limit as $r \to 0$:

$$|D\varphi_0(y)|^{\frac{ns}{s+1}} = \lim_{r\to 0}\frac{1}{|B(y,r)|}\int\limits_{B(y,r)} |D\varphi_0(x)|^{\frac{ns}{s+1}}\,dx \\ \leq \left(\lim_{r\to 0}\frac{1}{|B(y,r)|}\int\limits_{B(y,r)} M^s(x)\,dx\right)^{\frac{1}{s+1}} \\ \times \left(\lim_{r\to 0}\frac{1}{|B(y,r)|}\int\limits_{B(y,r)} J(x,\varphi_0)\,dx\right)^{\frac{s}{s+1}} \\ = M^s(y)^{\frac{1}{s+1}} J(y,\varphi_0)^{\frac{s}{s+1}}$$

for almost all $y \in \Omega$.

Thus, the pointwise inequality

$$|D\varphi_0(x)|^{\frac{ns}{s+1}} \le M^s(x)^{\frac{1}{s+1}} J(x,\varphi_0)^{\frac{s}{s+1}} \text{ for almost all } x \in \Omega$$

holds. It implies the assertion of the lemma:

$$\frac{|D\varphi_0(x)|^n}{J(x,\varphi_0)} \le M(x) \text{ for almost all } x \in \Omega.$$

□

4.2.8 Semicontinuity of the functional.

In order to complete the proof, it remains to verify that

$$\int\limits_{\Omega} W(x, D\varphi_0)\,dx \le \varliminf_{k\to\infty} \int\limits_{\Omega} W(x, D\varphi_k)\,dx. \tag{4.7}$$

If the right-hand side equals ∞ then the inequality is obvious. If the right-hand side is finite then there exists a subsequence φ_m for which the sequence $\left\{\int\limits_{\Omega} W(x, D\varphi_m)\,dx\right\}$ converges.

Using the weak convergence (4.3) and Mazur theorem, we see that for each m we can find an integer $j(m) \ge m$ and real numbers $\mu_m^t > 0$ for $m \le t \le j(m)$ such that

$$\sum_{t=m}^{j(m)} \mu_m^t = 1,$$

and in $L_n(\Omega) \times L_{\frac{n}{n-1}}(\Omega) \times L_r(\Omega)$ we have

$$D_m = \sum_{t=m}^{j(m)} \mu_m^t (D\varphi_t, \operatorname{Adj} D\varphi_t, J(\varphi_t)) \xrightarrow[m\to\infty]{} (D\varphi_0, \operatorname{Adj} D\varphi_0, J(\cdot,\varphi_0)).$$

Consequently, there exists a subsequence $\{D_l\}$ converging almost everywhere in Ω.

Since G satisfies Carathéodory conditions, it follows that $G(x,\cdot)$ is continuous for almost all $x \in \Omega$ and

$$W(x, D\varphi_0) = G(x, D\varphi_0, \operatorname{Adj} D\varphi_0, J(\cdot,\varphi_0)) \\ = \lim_{l\to\infty} G\Big(x, \sum_{t=l}^{j(l)} \mu_l^t (D\varphi_t, \operatorname{Adj} D\varphi_t, J(\cdot,\varphi_t))\Big).$$

Applying Fatou's lemma and the convexity of G, we arrive at

$$\int_\Omega W(x, D\varphi_0)\,dx \le \varliminf_{l\to\infty} \int_\Omega G\Big(x, \sum_{t=l}^{j(l)} \mu_l^t (D\varphi_t, \operatorname{Adj} D\varphi_t, J(\cdot,\varphi_t))\Big)\,dx \\ \le \varliminf_{l\to\infty} \sum_{t=l}^{j(l)} \mu_l^t \int_\Omega G(x, D\varphi_t, \operatorname{Adj} D\varphi_t, J(\cdot,\varphi_t))\,dx = \lim_{l\to\infty} W(x, D\varphi_l).$$

This justifies (4.7).

5 Examples

As our first example consider an Ogden material with the stored-energy function W_1 of the form

$$W_1(F) = a\operatorname{tr}(F^TF)^{\frac{p}{2}} + b\operatorname{tr}\operatorname{Adj}(F^TF)^{\frac{q}{2}} + c(\det F)^r + d(\det F)^{-s}, \qquad (5.1)$$

where $a > 0$, $b > 0$, $c > 0$, $d > 0$, $p > 3$, $q > 3$, $r > 1$, and $s > \frac{2q}{q-3}$. Then $W_1(F)$ is polyconvex and the coercivity inequality holds [7, Theorem 4.9-2]:

$$W_1(F) \ge \alpha\big(\|F\|^p + \|\operatorname{Adj} F\|^q\big) + c(\det F)^r + d(\det F)^{-s}.$$

Assume also that boundary conditions on the displacements are specified, and furthermore, $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ is a homeomorphism. We have to solve the minimization problem

$$I_1(\varphi_b) = \inf_{\varphi\in\mathcal{A}_B} I_1(\varphi),$$

where $I_1(\varphi) = \int_\Omega W_1(D\varphi(x))\,dx$ and the class of admissible deformations $\mathcal{A}_B$ is defined in (1.3). The result of John Ball [3] ensures that there exists at

least one solution $\varphi_B \in \mathcal{A}_B$ to this problem, which in addition is a homeomorphism.

On the other hand, for the functions of the form (5.1) Theorem 4.1 holds. Indeed, $W_1(F)$ is polyconvex and satisfies

$$W_1(F) \geq \alpha \|F\|^n + c(\det F)^r - \alpha,$$

where α plays the role of the function $h(x)$ of (4.1). When we consider the same boundary conditions $\overline{\varphi} : \overline{\Omega} \to \overline{\Omega'}$ and have to solve the minimization problem

$$I_1(\varphi_0) = \inf_{\varphi \in \mathcal{A}} I_1(\varphi)$$

in the class of admissible deformations $\mathcal{A}$ defined by (1.7), Theorem 4.1 yields a solution $\varphi_0 \in \mathcal{A}$ which is a homeomorphism.

This example shows that in those problems to which Ball's theorem applies the hypotheses of Theorem 4.1 are fulfilled; consequently, we can consider admissible deformations of class $\mathcal{A}$.

Let us discuss another example. Here the stored-energy function is of the form

$$W_2(F) = a\,\mathrm{tr}(F^T F)^{\frac{3}{2}} + c(\det F)^r.$$

This function is polyconvex and satisfies

$$W_2(F) \geq \alpha \|F\|^3 + c(\det F)^r,$$

but violates the inequality of the form (1.4). Moreover, $W_2(F)$ violates the asymptotic condition

$$W_2(x, F) \longrightarrow \infty \text{ as } \det F \longrightarrow 0_+,$$

which plays an important role in [3, 4] and other articles.

Nevertheless, for the stored-energy function W_2 there exists a solution to the minimization problem $I_2(\varphi_0) = \inf\limits_{\varphi \in \mathcal{A}} I_2(\varphi)$, where $I_2(\varphi) = \int\limits_{\Omega} W_2(D\varphi(x))\,dx$, which is a homeomorphism provided that a homeomorphism $\overline{\varphi}$ is prescribed on the boundary.

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